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Representation Functions of the Group of Motions of Clifford Space

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Another set of invariant operators of SU_n are the familiar quantities

$$N_r = \sum_{\substack{all\\indices}}^{n} \underbrace{E_{ij}E_{jk} \cdots E_{rs}E_{si}}_{r \text{ factors}}, \tag{4.17}$$

where the summation is now over all i, j, etc., and

$$E_{ii} = u_i \cdot H. \tag{4.18}$$

The eigenvalues $\tilde{N}_r$ are[6]

$$\tilde{N}_r = \sum_{\substack{all\\indices}}^{n} \underbrace{[\delta_{ik}u_k \cdot M + W_i \cdot u_k][\delta_{kr}u_r \cdot M + W_k \cdot u_r] \times \cdots [\delta_{ts}u_s \cdot M + W_t \cdot u_s]}_{r-1 \text{ factors}}(u_s \cdot M), \tag{4.19}$$

[6] A. M. Perelomov and V. S. Popov, JETP Letters 1, 15 (1965); see also F. Halbwachs, "Invariants Fondamentaux des Groups SU_n et $SU(n,1)$," preprint (1965).

where

$$W_i = \sum_{i<j} r_{ij} = (2n)^{-\frac{1}{2}}[(n-i)u_i] + \pi_i. \tag{4.20}$$

This expression may be rearranged to give

$$\tilde{N}_r = \sum_{\substack{all\\indices}}^{n} \underbrace{C_{ik}C_{kr} \cdots C_{ts}}_{r-1 \text{ factors}}(u_s \cdot M),$$

where

$$C_{ik} = \delta_{ik}\, u_k \cdot L + W_{ik}$$

and

$$W_{ik} = (2n)^{-\frac{1}{2}}[\tfrac{1}{2}(n+1)\,\delta_{ik} - \theta(k-i)].$$

The independence argument given above may be applied to the $\tilde{N}_r$ to establish them as fundamental invariants.

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Representation Functions of the Group of Motions of Clifford Space*

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We find the representation functions of the group of motions of the three-dimensional Clifford space and of the three-dimensional Einstein space. These functions are generalizations of spherical and cylindrical waves of three-dimensional Euclidian space and reduce to these familiar functions in the Euclidian limit. The generalization of the plane wave is also found.

1. INTRODUCTION

PHYSICAL space is often characterized as a three-dimensional metric space of absolute parallelism with positive definite symmetric metric and zero torsion. This space has an associated group of motions, that is, a group of coordinate transformations that leave the metric (and also the connection) invariant. This group is a six-parameter group usually decomposed into three displacements and three rotations.

A generalization of the above space may be made by allowing the torsion to be nonzero, but uniform.

This space still has a six-parameter group of motions with transformations corresponding to displacements and rotations. In this paper we find the eigenfunctions of the operators associated with these transformations and show their relationship to the usual eigenfunctions of Euclidian space, i.e., the spherical, cylindrical, and plane waves.

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2. SPECIFICATION OF THE SPACE

The generalized space S considered here may be characterized as follows[1,2]:

$$L^{\mu}_{\alpha\beta\gamma}(+) = 0, \tag{2.1}$$

$$\Omega^{\mu}_{\alpha\beta|\gamma^+} = 0, \tag{2.2}$$

$$g_{\alpha\beta|\gamma^+} = 0, \tag{2.3}$$

where[3] $L^{\mu}_{\alpha\beta\gamma}(+)$, $\Omega^{\mu}_{\alpha\beta}$, and $g_{\alpha\beta}$ are tensors representing the curvature, torsion, and metric. The $|^{\pm}$ sign appearing above means the covariant derivative with respect to the $(+)$ connection $L^{\mu}_{\alpha\beta} \equiv L^{\mu}_{\alpha\beta}(+)$ and

[1] L. P. Eisenhart, *Continuous Groups of Transformations* (Dover Publications Inc., New York, 1961), pp. 51 and 231.

[2] E. Cartan and J. A. Schouten, Akad. van Wetens, Amsterdam, Proc. 29, 803 (1926).

[3] Latin and Greek indices run from 1 to 3 and the summation convention is used.

the $(-)$ connection $L^{\mu}_{\alpha\beta}(-)$, where

$$L^{\mu}_{\alpha\beta} \equiv L^{\mu}_{\alpha\beta}(+) \equiv \Gamma^{\mu}_{\alpha\beta} + \Omega^{\mu}_{\alpha\beta}, \tag{2.4}$$

$$L^{\mu}_{\alpha\beta}(-) \equiv \Gamma^{\mu}_{\alpha\beta} - \Omega^{\mu}_{\alpha\beta}, \tag{2.5}$$

$\Gamma^{\mu}_{\alpha\beta}$ and $\Omega^{\mu}_{\alpha\beta}$ are, respectively, symmetric and antisymmetric in α and β.

$$\Gamma^{\mu}_{\alpha\beta} = \Gamma^{\mu}_{\beta\alpha}, \tag{2.6}$$

$$\Omega^{\mu}_{\alpha\beta} = -\Omega^{\mu}_{\beta\alpha}. \tag{2.7}$$

Explicitly, the $(\pm)$-covariant derivatives and a sometimes used (0)-covariant derivative are given below where A_{μ} is an arbitrary covariant vector.

$$A_{\mu|\nu^+} \equiv \partial_{\nu}A_{\mu} - L^{\sigma}_{\mu\nu}(+)A_{\sigma}, \tag{2.8}$$

$$A_{\mu|\nu^-} \equiv \partial_{\nu}A_{\mu} - L^{\sigma}_{\mu\nu}(-)A_{\sigma}, \tag{2.9}$$

$$A_{\mu|\nu^\circ} \equiv \partial_{\nu}A_{\mu} - \Gamma^{\sigma}_{\mu\nu}A_{\sigma}. \tag{2.10}$$

From (2.3) it follows that $\Omega_{\mu\alpha\beta}$ is completely antisymmetric. If the torsion is taken to be zero, S reduces to the usual Euclidian space.

3. ADDITIONAL PROPERTIES OF S

The properties of the space defined by the $(-)$ connection are the same as the space defined by the $(+)$ connection; that is,

$$L^{\mu}_{\alpha\beta\gamma}(-) = 0 \tag{3.1}$$

and

$$\Omega^{\mu}_{\alpha\beta|\gamma^-} = 0. \tag{3.2}$$

Also, the space defined by $g_{\alpha\beta}$ and $\Gamma^{\mu}_{\alpha\beta}$ is an Einstein space, that is,

$$g_{\alpha\beta|\gamma^\circ} = 0 \tag{3.3}$$

and

$$B_{\alpha\beta} = -2Kg_{\alpha\beta}, \tag{3.4}$$

where $B_{\alpha\beta}$ is the contracted curvature tensor formed from $\Gamma^{\mu}_{\alpha\beta}$. Consequently,

$$\Gamma^{\mu}_{\alpha\beta} = \left\{ {\mu \atop \alpha\beta} \right\}, \tag{3.5}$$

where $\left\{ {\mu \atop \alpha\beta} \right\}$ is the Christoffel symbol.

4. ENNUPLE FIELDS

Since S has the property $L^{\mu}_{\alpha\beta\gamma}(\pm) = 0$, the two equations $\lambda_{\mu|\nu^\pm} = 0$ may be integrated to give two vector fields which are individually everywhere parallel to themselves.

If we take three vector fields $\lambda^i_{\mu}(+)$ which are orthogonal at a point, then they are orthogonal everywhere. We think of these three vector fields as defining a basis at every point of S, the bases at different points being related by $(+)$-parallel transfer.

The same considerations apply to three vector fields orthogonal at a point and satisfying $\lambda^i_{\mu|\nu^-} = 0$. They are orthogonal everywhere and are related by $(-)$-parallel transfer.

We define an ennuple fundamental, or metric tensor $\bar{g}_{ij}$ by

$$\begin{aligned} \bar{g}_{ij} &= 1 \quad \text{if} \quad i = j, \\ \bar{g}_{ij} &= 0 \quad \text{if} \quad i \neq j, \end{aligned} \tag{4.1}$$

and the ennuple tensor $\bar{g}^{ij}$ by $\bar{g}^{ij}\bar{g}_{jk} = \delta^i{}_k$.

We then associate with the vectors λ^i_{μ} the vectors

$$\lambda_{i\mu} = \bar{g}_{ij}\lambda^j_{\mu}, \tag{4.2}$$

$$\lambda^{i\mu} = g^{\mu\nu}\lambda^i_{\nu}, \tag{4.3}$$

and

$$\lambda^{\mu}_i = \bar{g}_{ij}g^{\mu\nu}\lambda^j_{\nu}. \tag{4.4}$$

5. GROUP OF MOTIONS

Consider the following six quantities:

$$X_i(\pm) \equiv \lambda^{\mu}_i(\pm)\,\partial_{\mu}. \tag{5.1}$$

Since the $\lambda^i_{\mu}(\pm)$ satisfy Killing's equation

$$\lambda^i_{\mu|\nu^\circ} + \lambda^i_{\nu|\mu^\circ} = 0, \tag{5.2}$$

the quantities (5.1) are the generators of coordinate transformations that leave the metric invariant. The commutation relations of these generators are as follows:

$$[X_i(+), X_j(+)] = C_{ij}{}^k(+)X_k(+), \tag{5.3a}$$

$$[X_i(-), X_j(-)] = C_{ij}{}^k(-)X_k(-), \tag{5.3b}$$

$$[X_i(+), X_j(-)] = 0, \tag{5.3c}$$

where

$$C_{ij}{}^k(\pm) = \pm 2\lambda^k_{\mu}(\pm)\lambda^{\alpha}_i(\pm)\lambda^{\beta}_j(\pm)\Omega^{\mu}_{\alpha\beta}. \tag{5.4}$$

The $C_{ij}{}^k(\pm)$ are constant scalars because they satisfy

$$C_{ij}{}^k(\pm)_{|\nu^\pm} = 0 = \partial_{\nu}\, C_{ij}{}^k(\pm). \tag{5.5}$$

They also satisfy the Jacobi relations and are thus generators of a Lie group.

The $C_{ij}{}^k(+)$ and $C_{ij}{}^k(-)$ are related by choosing the $\lambda^i_{\mu}(+)$ and $\lambda^i_{\mu}(-)$ to coincide at one point. Then

$$C_{ij}{}^k(+) = -C_{ij}{}^k(-). \tag{5.6}$$

The quantity

$$C_{ijk}(\pm) = C_{ij}{}^l(\pm)\bar{g}_{lk} \tag{5.7}$$

is totally antisymmetric as a result of the antisymmetry of $\Omega_{\alpha\beta\gamma}$. Consequently, we set

$$C_{ij}{}^k(+) = 2K^{\frac{1}{2}}\epsilon_{ij}{}^k, \tag{5.8a}$$

$$C_{ij}{}^k(-) = -2K^{\frac{1}{2}}\epsilon_{ij}{}^k. \tag{5.8b}$$

The invariance of $\Omega^{\mu}_{\alpha\beta}$ under (5.1) is a consequence of $\Omega_{\mu\alpha\beta} \sim g^{\frac{1}{2}}\epsilon_{\mu\alpha\beta}$.

Since $\Gamma^{\mu}_{\alpha\beta}$ is derived from the metric, it is apparent that the connection is invariant under (5.1) and thus they constitute the generators of the group of motions of S.

If we introduce the operators

$$\tilde{X}_i(+) = \frac{1}{2iK^{\frac{1}{2}}} X_i(+), \qquad i \neq 1, \tag{5.9a}$$

$$\tilde{X}_1(+) = -\frac{1}{2iK^{\frac{1}{2}}} X_1(+), \tag{5.9b}$$

$$\tilde{X}_i(-) = \frac{1}{2iK^{\frac{1}{2}}} X_i(-), \tag{5.9c}$$

then the commutation relations take the usual form for the four-dimensional orthogonal group.

The relationship between the group of motions of S and 0_4 may be seen in another way. As regards the metric alone, the space S is that of the surface of a four-dimensional sphere since $g_{\alpha\beta}$ describes an Einstein space. Transformations of this sphere that leave the metric of its surface invariant are the rotations of the sphere about its center. These transformations belong to 0_4.

6. REPRESENTATION FUNCTIONS

We now wish to find representation functions of the group of motions on the space S. The representation functions of 0_4 on the four-dimensional Euclidian space are functions of three independent coordinates, the fourth coordinate being restricted by the condition that the radius of the sphere remain invariant. These functions are the hyperspherical harmonics and they constitute representation functions of the group of motions of S.

In general, 0_4 representations are labeled by the eigenvalues of two Casimir operators associated with the two invariant 0_3 subgroups, namely $j(j+1)$ and $j'(j'+1)$. From functions on a four-dimensional Euclidian space, only certain representations of 0_4 may be found, namely, those for which the Casimir operator eigenvalues are identical or for which $j = j'$.

In terms of the $X_i(\pm)$, the two Casimir operators are

$$\begin{aligned} \bar{g}^{ij}X_i(+)X_j(+) &\equiv X^2(+), \\ \bar{g}^{ij}X_i(-)X_j(-) &\equiv X^2(-), \end{aligned} \tag{6.1}$$

and[4] these operators are seen to be identical when expressed in terms of the λ's. Also X^2 is the Laplacian in S

$$X^2(+) = X^2(-) = \Delta = \bar{g}^{\frac{1}{2}}\,\partial_\nu(g^{\frac{1}{2}}g^{\mu\nu}\,\partial_\mu). \tag{6.2}$$

The form of the representation functions depends upon the choice of the set of eigenvalues used as labels. We are guided by the desire to make a connection with the well-known representation functions of three-dimensional Euclidian space. Consequently, the following linear and bilinear combinations of the $X_i(\pm)$ are of interest.

$$P_i = (1/i)X_i(+), \tag{6.3a}$$

$$L_i = (1/2iK^{\frac{1}{2}})[X_i(-) - X_i(+)], \tag{6.3b}$$

$$L^2 = \bar{g}^{ij}L_iL_j, \tag{6.3c}$$

$$P^2 = \bar{g}^{ij}P_iP_j = -X^2. \tag{6.3d}$$

These operators have the following commutation relations:

$$\begin{aligned} &[P_i, L_j] = i\epsilon_{ijk}P_k, && [P_i, P_j] = 2K^{\frac{1}{2}}i\epsilon_{ijk}P_k, \\ &[L_i, L_j] = i\epsilon_{ijk}L_k, && [P^2, L_i] = 0, \\ &[P^2, P_i] = 0, && [L^2, L_i] = 0, \\ &[L^2, P_i] = i\epsilon_{ijk}\{P_j, L_k\}, && [L^2, P^2] = 0. \end{aligned} \tag{6.4}$$

The operators have the property that in the limit of $K \to 0$, i.e., in the Euclidian limit, their commutation relations become those for P_i, L_i, P^2, L^2, where P_i and L_i are the usual momentum and angular momentum operators.

We are concerned with the representation functions for which the following sets of operators are diagonal

$$P^2, L^2, L_3 \tag{6.5a}$$

and

$$P^2, P_3, L_3. \tag{6.5b}$$

The functions associated with the first set correspond in Euclidian space to $j_l(kr)Y_{lm}(\theta, \phi)$. The eigenvalues of P^2, L^2, L_3 are respectively k^2, $l(l+1)$, and m.

The functions associated with the second set correspond in Euclidian space to $J_m[(k^2 - k_3^2)^{\frac{1}{2}}\rho]e^{ik_3x^3}e^{im\phi}$ with eigenvalues k^2, k_3, and m for the three operators.

7. RIEMANNIAN COORDINATE SYSTEM

We begin the study of the various differential operators of (6.5) in the Riemannian coordinate system. The vector fields $\lambda^i_\mu(\pm)$ may be expressed in this system in the following way[1]:

$$\begin{aligned} \lambda^i_\mu(\pm) &= \delta^i_\mu + \tfrac{1}{2}C_{\mu\alpha_1}{}^i(\pm)x^{\alpha_1} + \cdots \\ &\quad + [1/(r+1)!]C_{\mu\alpha_1}{}^{\beta_1}C_{\beta_1\alpha_2}{}^{\beta_2}\cdots C_{\beta_{r-1}}{}^i{}_{\alpha_r}x^{\alpha_1}\cdots x^{\alpha_r} \\ &= \delta^i_\mu\frac{\sin\omega}{\omega} + \omega^\mu\omega^i\,\frac{\omega - \sin\omega}{\omega^3} + \omega^\alpha\epsilon_{i\mu\alpha}\,\frac{1-\cos\omega}{\omega}, \end{aligned} \tag{7.1}$$

[4] The Killing form is identical to $\bar{g}_{ij}$.

where dimensionless coordinates ω^μ have been introduced

$$\omega^\mu = \pm 2K^{\frac{1}{2}} x^\mu \tag{7.2a}$$

and

$$\omega^2 = \sum_{\mu=1}^{3} (\omega^\mu)^2. \tag{7.2b}$$

It is seen that in the limit $K \to 0$, the λ_μ^i go to δ_μ^i. The parallel vector fields of Euclidian space are the Cartesian coordinate axes.

The other quantities of interest are

$$g_{\mu\nu} = \bar{g}_{ij}\lambda_\mu^i\lambda_\nu^j = \delta_\nu^\mu \frac{2(1-\cos\omega)}{\omega^2} + \frac{\omega^\mu\omega^\nu}{\omega^4}(\omega^2 - 2 + 2\cos\omega), \tag{7.3}$$

$$g^{\mu\nu} = \delta_\nu^\mu \frac{\omega^2}{2(1-\cos\omega)} + \omega^\mu\omega^\nu \frac{(2 - 2\cos\omega - \omega^2)}{2\omega^2(1-\cos\omega)}, \tag{7.4}$$

$$\lambda_i^\mu(\pm) = \delta_i^\mu \frac{\omega\sin\omega}{2(1-\cos\omega)} + \frac{\omega^\mu\omega^i(2 - 2\cos\omega - \omega\sin\omega)}{2\omega^2(1-\cos\omega)} + \frac{\omega^\alpha}{2}\epsilon_{\mu\alpha i}, \tag{7.5}$$

$$P_i = \frac{1}{i}\left[\frac{\omega\sin\omega}{2(1-\cos\omega)}\frac{\partial}{\partial x^i} + \frac{\omega^\mu\omega^i(2-2\cos\omega-\omega\sin\omega)}{2\omega^2(1-\cos\omega)}\frac{\partial}{\partial x^\mu} + K^{\frac{1}{2}}x^\alpha\epsilon_{\mu\alpha i}\frac{\partial}{\partial x^\mu}\right], \tag{7.6}$$

$$L_i = \frac{1}{i}x^\alpha\epsilon_{\mu\alpha i}(\partial/\partial x^\mu) = (1/i)(x^j\,\partial_k - x^k\,\partial_j). \tag{7.7}$$

It is easily seen that as $K \to 0$, $P_i \to (1/i)(\partial/\partial x^i)$. The L_i already have the form of the angular momentum operator in Cartesian coordinates.

8. SPHERICAL WAVES

To obtain the generalization of spherical waves in S, we transform from the ω^μ system to "polar" coordinates

$$\omega = \left[\sum_{\mu=1}^{3}(\omega^\mu)^2\right]^{\frac{1}{2}},$$
$$\tan\theta = [(\omega^1)^2 + (\omega^2)^2]^{\frac{1}{2}}/(\omega^3), \tag{8.1}$$
$$\tan\phi = (\omega^2)/(\omega^1).$$

The coordinate ω is dimensionless and equals $2K^{\frac{1}{2}}r$, where r^2 is the usual sum $\sum_{\mu=1}^{3}(x^\mu)^2$. The ranges of the coordinates are

$$0 \le \omega \le 2\pi,$$
$$0 \le \theta \le \pi, \tag{8.2}$$
$$0 \le \phi \le 2\pi.$$

The Laplacian in this coordinate system takes the form,

$$\Delta = 4K\left\{\frac{\partial^2}{\partial\omega^2} + \cot\frac{\omega}{2}\frac{\partial}{\partial\omega} + \frac{1}{4\sin^2\frac{1}{2}\omega} \times\left[\frac{\partial^2}{\partial\theta^2} + \cot\theta\frac{\partial}{\partial\theta} + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right]\right\}. \tag{8.3}$$

The equation

$$P^2\psi = -X^2\psi = -\Delta\psi = Kj(j+1)\psi \tag{8.4}$$

is solved by separation of variables to give

$$\psi_s = \bar{\jmath}_l(j, \omega)Y_{lm}(\theta, \phi), \tag{8.5}$$

where the $Y_{lm}(\theta, \phi)$ are the usual spherical harmonics and the $\bar{\jmath}_l(j, \omega)$ satisfy

$$\left\{\frac{d^2}{d\omega^2} + \cot\frac{\omega}{2}\frac{d}{d\omega} + \left[j(j+1) - \frac{l(l+1)}{4\sin^2\frac{1}{2}\omega}\right]\right\}\bar{\jmath}_l(j,\omega) = 0 \tag{8.6}$$

with solutions[5]

$$\bar{\jmath}_l(j,\omega) = \left[\frac{(n^2 - l^2 - 1)!}{4\pi(n^2)!}\right]^{\frac{1}{2}} \times \sin^l\frac{\omega}{2}\, d^{l+1}\frac{\cos\frac{1}{2}n\omega}{d(\cos\frac{1}{2}\omega)^{l+1}}, \tag{8.7}$$

where $n = 2j + 1$. The metric is

$$g_{\mu\nu} = \frac{1}{4K}\begin{pmatrix} 1 & & \\ & 4\sin^2(\frac{1}{2}\omega) & \\ & & 4\sin^2(\frac{1}{2}\omega)\ \sin^2\theta \end{pmatrix}. \tag{8.8}$$

The invariant volume element is

$$d\tau = g^{\frac{1}{2}}\,d\omega\,d\theta\,d\phi = \tfrac{1}{2}R^3\sin^2(\tfrac{1}{2}\omega)\sin\theta\,d\omega\,d\theta\,d\phi \tag{8.9}$$

and the normalization of the ψ_s over the invariant volume is taken to be unity. The operators L^2 and L_3 have the forms

$$L^2 = -\left(\frac{\partial^2}{\partial\theta^2} + \cot\theta\frac{\partial}{\partial\theta} + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right), \tag{8.10a}$$

$$L_3 = (1/i)(\partial/\partial\phi), \tag{8.10b}$$

and with P^2 have the eigenvalues

$$P^2 - 4Kj(j+1),$$
$$L^2 - l(l+1), \tag{8.11}$$
$$L^3 - m,$$

[5] These solutions are related to solutions of the hydrogen atom in momentum space; see V. Fock [Z. Physik **98**, 145 (1935)]. Schrödinger discusses these solutions in his series of articles on eigenvalue problems in a hypersphere [E. Schrödinger, Commen. Pont. Acad. Sci. **2**, 321 (1938); Proc. Roy. Irish Acad. **XLVI**, Sec. A, **9**, 25 (1940)].

where j takes on nonnegative integer and half-integer values and l and m range from 0 to $2j$ and $-l$ to l, respectively, in integer steps.

The behavior of ψ_s for small K is best seen via the differential equation that the $\tilde{j}_l(j,\omega)$ functions satisfy, the Y_{lm} being unaffected in the limit $K \to 0$.

We expand (8.6) for small values of ω, keeping only lowest-order terms in ω but allowing for large j. We obtain

$$\left\{\omega^2 \frac{d^2}{d\omega^2} + 2\omega \frac{d}{d\omega} + [\omega^2 j^2 - l(l+1)]\right\}\tilde{j}_l(j,\omega) = 0 \tag{8.12}$$

with solutions $j_l(j\omega)$ or $j_l(kr)$, where $k = 2K^{\frac{1}{2}}j$.

Thus the usual momentum $k^2 = 4Kj^2$ is seen to correspond to the eigenvalue of P^2 in the limit $K \to 0$, $K^{\frac{1}{2}}j$ remaining finite.

$$P^2\psi_s = 4Kj(j+1)\psi_s \underset{j\to\infty}{\overset{K\to 0}{\longrightarrow}} 4Kj^2\psi_s = k^2\psi_s. \tag{8.13}$$

The interpretation of ψ_s as a representation function of 0_4 is made by identifying ω, θ, and ϕ with the angular coordinates of a sphere in a four-dimensional Euclidian space. If y^μ are the coordinates of a unit sphere, then

$$y^1 = \sin \tfrac{1}{2}\omega \sin\theta \cos\phi, \qquad y^2 = \sin\tfrac{1}{2}\omega \sin\theta \sin\phi,$$
$$y^3 = \sin\tfrac{1}{2}\omega \cos\theta, \qquad y^4 = \cos\tfrac{1}{2}\omega, \tag{8.14}$$

where

$$\sum_{\mu=1}^{4} (y^\mu)^2 = 1. \tag{8.15}$$

9. CYLINDRICAL WAVES

The usual method of finding the cylindrical waves of Euclidian space involves a transformation to cylindrical coordinates and a separation of the Helmholtz equation in that system. The transformation from spherical coordinates is

$$x^3 = r\cos\theta, \qquad \rho = r\sin\theta, \qquad \phi = \phi. \tag{9.1}$$

The generalization of this transformation is seen as follows. On a unit three-dimensional sphere draw a meridian M through an arbitrary origin or pole. A point A in one hemisphere may be labeled by its longitude θ with respect to M and polar distance $\frac{1}{2}\omega$ or, by the shortest distance $\frac{1}{2}\beta$ from A to M, which is via the great circle C through A and perpendicular to M, and the polar distance $\frac{1}{2}\omega_3$ of the intersection M and C. To make the labeling unique, points of $\theta < \frac{1}{2}\pi$ have $\omega_3 > 0$ and points of $\theta > \frac{1}{2}\pi$ have $\omega_3 < 0$. β is seen to range from 0 to π and ω_3 from -2π to 2π. The relationship between the coordinate systems is (see Fig. 1)

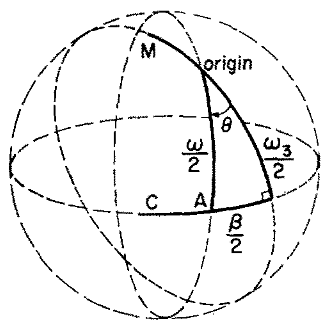

Fig. 1. Relationship between "polar" and "cylindrical" coordinates.

$$\sin\tfrac{1}{2}\beta = \sin\tfrac{1}{2}\omega \sin\theta,$$
$$\tan\tfrac{1}{2}\omega_3 = \tan\tfrac{1}{2}\omega\cos\theta, \tag{9.2}$$
$$\phi = \phi.$$

Equations (9.2) reduce to (9.1) in the limit $K \to 0$. Here $\beta = 2K^{\frac{1}{2}}\rho$ and $\omega_3 = \omega^3 = 2K^{\frac{1}{2}}x^3$.

The equation

$$[\Delta + 4Kj(j+1)]\psi = 0 \tag{9.3}$$

now takes the form

$$\left[\frac{\partial^2}{\partial\beta^2} + \cot\beta\frac{\partial}{\partial\beta} + \frac{1}{\cos^2(\frac{1}{2}\beta)}\frac{\partial^2}{\partial\omega_3^2} + \frac{1}{4\sin^2(\frac{1}{2}\beta)}\frac{\partial^2}{\partial\phi^2} + j(j+1)\right]\psi_c = 0 \tag{9.4}$$

with solutions

$$\psi_c = \mathcal{J}_m(j,s;\beta)e^{is\omega_3}e^{im\phi}, \tag{9.5}$$

where $\mathcal{J}_m(j,s;\beta)$ satisfies

$$\left\{\frac{d^2}{d\beta^2} + \cot\beta\frac{d}{d\beta} - \frac{1}{\sin^2\beta}\left[2s^2 + \frac{m^2}{2} - \cos\beta\left(2s^2 - \frac{m^2}{2}\right)\right] + j(j+1)\right\}\mathcal{J} = 0. \tag{9.6}$$

With the substitution

$$s = \tfrac{1}{2}(\mu + \nu), \tag{9.7a}$$
$$m = -\mu + \nu, \tag{9.7b}$$

Eq. (9.6) takes the form

$$[(d^2/d\beta^2) + \cot\beta - (1/\sin^2\beta) \times (\mu^2 + \nu^2 - 2\mu\nu\cos\beta) + j(j+1)]\mathcal{J} = 0 \tag{9.8}$$

with solutions

$$J_m(j,s;\beta) \sim d^j_{\mu\nu}(\beta), \tag{9.9}$$

where the $d^j_{\mu\nu}(\beta)$ are the familiar functions of the finite representation theory of 0_3.

The complete solutions may be put into the form of the $D^j_{\mu\nu}(\alpha,\beta,\gamma)$ functions as follows:

$$\begin{aligned}\psi_c &= \mathcal{J}_m(j,s;\beta)e^{is\omega_3}e^{im\phi}\\ &= (-1)^{\mu+\nu}Nd^j_{\mu\nu}(\beta)e^{i\mu\alpha}e^{i\nu\gamma}\\ &= (-1)^{\mu+\nu}ND^j_{\mu\nu}(\alpha,\beta,\gamma),\end{aligned} \tag{9.10}$$

where

$$\gamma = \tfrac{1}{2}\omega_3 + \phi + \pi, \tag{9.11a}$$

$$\alpha = \tfrac{1}{2}\omega_3 - \phi + 3\pi, \tag{9.11b}$$

and

$$0 \leq \alpha \leq 4\pi, \quad 0 \leq \gamma \leq 4\pi. \tag{9.12}$$

N is a normalization factor which is now computed. The metric takes the form

$$g_{\mu\nu} = \frac{1}{4K}\begin{bmatrix} 1 & & \\ & \cos^2(\tfrac{1}{2}\beta) & \\ & & 4\sin^2(\tfrac{1}{2}\beta) \end{bmatrix} \tag{9.13}$$

with the invariant volume element given by (here $R^2 = K^{-1}$)

$$d\tau = g^{\frac{1}{2}}\, d\beta\, d\omega_3\, d\phi = \tfrac{1}{8}R^3 \sin\beta\, d\beta\, d\omega_3\, d\phi. \tag{9.14}$$

We have

$$\begin{aligned} \delta_{jj'}\,\delta_{ss'}\,\delta_{mm'} &= \frac{R^3}{8}|N|^2 \int_0^{\pi}\int_{-2\pi}^{2\pi}\int_0^{2\pi} \sin\beta (d^{j'}_{\mu'\nu'} e^{is'\omega_3} e^{im'\phi})^* \\ &\quad\times (d^j_{\mu\nu} e^{is\omega_3} e^{im\phi})\, d\beta\, d\omega_3\, d\phi \\ &= R^3 |N|^2 \pi^2 \delta_{ss'}\,\delta_{mm'} \int_0^{\pi} \sin\beta d^{*j'}_{\mu\nu}\, d^j_{\mu\nu}\, d\beta \\ &= \frac{2\pi^2 R^3 |N|^2}{2j+1}\delta_{jj'}\,\delta_{ss'}\,\delta_{mm'} \end{aligned} \tag{9.15}$$

or

$$N = \left(\frac{2j+1}{2\pi^2 R^3}\right)^{\frac{1}{2}} = \left(\frac{2j+1}{V}\right)^{\frac{1}{2}}, \tag{9.16}$$

where V is the total volume of S. The last integration is based on the orthogonality of the $d^j_{\mu\nu}$.

The operators P^2, P_3, L_3 have eigenvalues $4Kj(j+1)$, $2K^{\frac{1}{2}}\mu = 2K^{\frac{1}{2}}(s - \frac{1}{2}m)$, and $-\mu + \nu = m$, where μ and ν have the ranges $-j \leq {}^{\mu}_{\nu} \leq j$ in integer steps.

The equation for $\tilde{J}$ may be expanded for small K, or β, keeping only lowest-order terms but allowing for large j and s.

We have from (9.6),

$$\left(\frac{d^2}{d\beta^2} + \frac{1}{\beta}\frac{d}{d\beta} + j^2 - s^2 - \frac{m^2}{\beta^2}\right)\tilde{J} = 0$$

or Bessel's equation with solutions $J_m[(j^2 - s^2)^{\frac{1}{2}}\beta]$. The argument is $(k^2 - k_3^2)^{\frac{1}{2}}\rho$, where $k^2 = 4j^2K$ and $k_3 = 2K^{\frac{1}{2}}s$.

10. PLANE WAVES

We are now in a position to see the generalization of the plane wave of flat space. The wave $J_m[(k^2 - k_3^2)^{\frac{1}{2}}\rho]e^{ik_3x^3}e^{im\phi}$ reduces to a plane wave in the three-direction when k_3 assumes its maximum value and when $m = 0$. The same two conditions when applied to the generalized cylindrical wave define a function which we call a generalized plane wave. Thus the plane wave is

$$\begin{aligned} \psi_P &= [(2j+1)/V]^{\frac{1}{2}}\, d^j_{jj}(\beta) e^{ij\omega_3} \\ &= [(2j+1)/V]^{\frac{1}{2}} \cos^{2j}(\tfrac{1}{2}\beta) e^{ij\omega_3}. \end{aligned} \tag{10.1}$$

When expressed in terms of the dimensional coordinates and expanded for large R, we have

$$\begin{aligned} \psi_P &= \left(\frac{2j+1}{V}\right)^{\frac{1}{2}}\left(1 - \sin^2\frac{\rho}{R}\right)^j e^{ik_3x^3} \\ &\to \left(\frac{k_3}{RV}\right)^{\frac{1}{2}} e^{ik_3x^3}\left(1 - \frac{k_3\rho^2}{2R} + \cdots\right). \end{aligned} \tag{10.2}$$

11. EINSTEIN SPACE

As shown earlier, the space defined by $g_{\alpha\beta}$ and $\Gamma^{\mu}_{\alpha\beta}$ is the Einstein space. Its group of motions is the same as that of $S^{(2)}$. Thus, the above functions are also representation functions of the group of motions of the Einstein space on the space itself.

In the case of S, K is a measure of the torsion, there being no curvature to the space. In the case of the Einstein space, K is a measure of the curvature of the space, the torsion being zero.

12. SUMMARY

The space described here is in several ways a convenient one to describe physical space. The observed Hubble effect may be given an interpretation in terms of either an expanding Einstein space or an expanding Clifford space. Also, coordinate reflections do not take the space into itself and consequently, linear equations involving the operators $X_i(\pm)$ are not invariant under parity operations.[6] Such a situation is interesting from the point of view of the description of processes in which parity is not conserved or is only partially conserved, such as in weak interactions.

The functions discussed in this paper are basis functions for a description of quantization in Clifford space. The scalar field theory generalizes to Clifford space with no difficulty.[7] Spinor quantization problems are now being investigated by the author.[8]

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