Derivation of the Spherical Law of Cosines and Sines using Rotation Matrices

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The Spherical Law of Cosines and Spherical Law of Sines, which relate the angles and sides of a spherical triangle in three dimensions, are derived using rotation matrices. As well as the usual two laws, new relationships between the angles and sides of a spherical triangle are derived.

1.

With a typical right handed x, y, z axis system, a rotation about the z-axis by an angle α is given by the matrix Z where

$$Z(\alpha) = \begin{bmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad (1.1)$$

The corresponding Y matrix representing a rotation of angle β about the y-axis is given by

$$Y(\beta) = \begin{bmatrix} \cos\beta & 0 & \sin\beta \\ 0 & 1 & 0 \\ -\sin\beta & 0 & \cos\beta \end{bmatrix} \qquad (1.2)$$

2.

Consider the spherical triangle in Figure 1, composed of portions of three great circles on the unit sphere. The three supplementary vertex angles α_1, α_2, and α_3 may be viewed as representing angular rotations about perpendiculars to the sphere at the three vertices. The three legs of the spherical triangle β_1, β_2 and β_3 may be viewed as representing angular rotations about perpendiculars through the centers of the three great circles that contain the legs.

If the origin of a rectangular coordinate system is placed at the center of the unit sphere such that the vertex associated with α_1 is on the z-axis of the coordinate system, and the leg β_3 lies in the x-z plane, then the following argument may be made. See Figure 1.
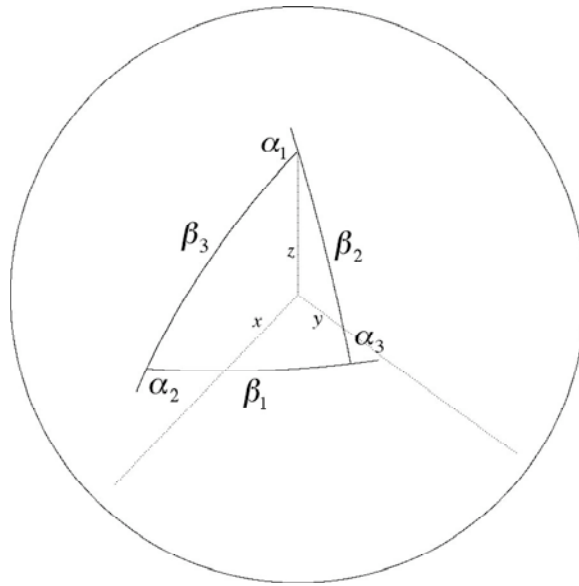
Figure 1.

3.

Consider a sequence of rotations of the sphere. The first rotation is performed about the z-axis by an angle α_1. This brings the side β_2 into the x-z plane. A second rotation is performed about the y-axis by β_2. This rotation brings the vertex associated with α_3 to the z-axis. A rotation about the z-axis by α_3 brings the side β_1 into the x-z plane. With a rotation about the y-axis by β_1, followed by a rotation about the z-axis by α_2, followed by a final rotation about the y-axis by β_3, the spherical triangle is returned to its original position. See Figure 2.

$Z(\alpha_1)$ $\qquad$ $Y(\beta_2)$ $\qquad$ $Z(\alpha_3)$

$Y(\beta_1)$ $\qquad$ $Z(\alpha_2)$ $\qquad$ $Y(\beta_3)$
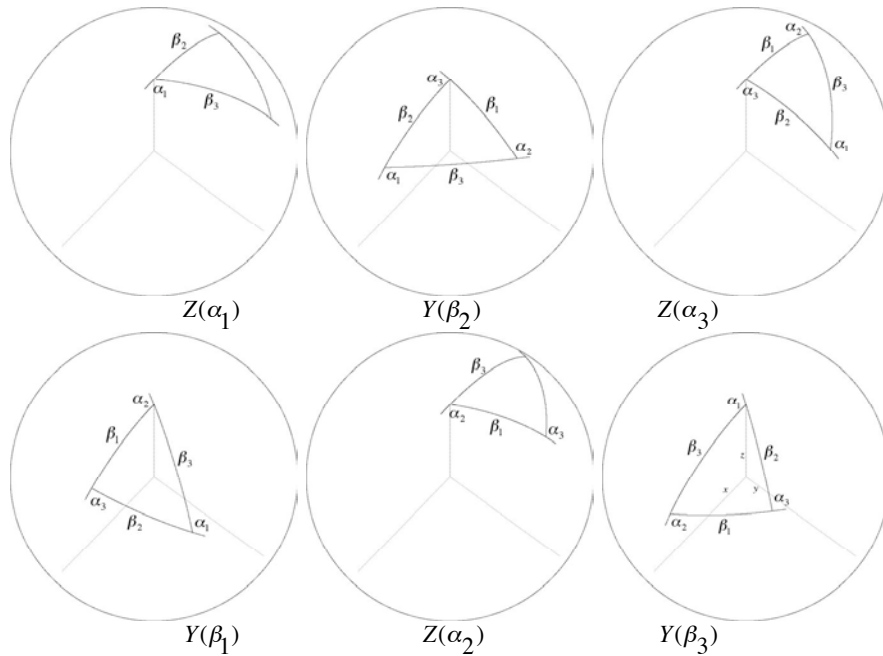
Figure 2.

The effect of these six rotations may be expressed as follows.

$$Y(\beta_3)Z(\alpha_2)Y(\beta_1)Z(\alpha_3)Y(\beta_2)Z(\alpha_1) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad (3.1)$$

An expansion of the left side of this equation is not interesting due to its complexity, but if the equation is rearranged, as follows, it becomes more manageable and interesting.

$$Y(\beta_3)Z(\alpha_2)Y(\beta_1) = Z(-\alpha_1)Y(-\beta_2)Z(-\alpha_3) \qquad (3.2)$$

Expansion of the left side and of the right side results in a matrix identity relating the nine elements of the left hand matrix with the nine elements of the right hand matrix.

4.

The nine identities are

11) $\cos\beta_3\cos\alpha_2\cos\beta_1 - \sin\beta_3\sin\beta_1 = \cos\alpha_1\cos\beta_2\cos\alpha_3 - \sin\alpha_1\sin\alpha_3$

12) $-\cos\beta_3\sin\alpha_2 = \cos\alpha_1\cos\beta_2\sin\alpha_3 + \sin\alpha_1\cos\alpha_3$

13) $\cos\beta_3\cos\alpha_2\sin\beta_1 + \sin\beta_3\cos\beta_1 = -\cos\alpha_1\sin\beta_2$

21) $\sin\alpha_2\cos\beta_1 = -\sin\alpha_1\cos\beta_2\cos\alpha_3 - \cos\alpha_1\sin\alpha_3$

22) $\cos\alpha_2 = -\sin\alpha_1\cos\beta_2\sin\alpha_3 + \cos\alpha_1\cos\alpha_3$ 　　　　　(4.1)

23) $\sin\alpha_2\sin\beta_1 = \sin\alpha_1\sin\beta_2$

31) $-\sin\beta_3\cos\alpha_2\sin\beta_1 + \cos\beta_3\cos\beta_1 = \sin\beta_2\cos\alpha_3$

32) $\sin\beta_3\sin\alpha_2 = \sin\beta_2\sin\alpha_3$

33) $-\sin\beta_3\cos\alpha_2\sin\beta_1 + \cos\beta_3\cos\beta_1 = \cos\beta_2$

Identities 23 and 32 are recognized as expressions that lead to the Spherical Law of Sines.

$$\frac{\sin\alpha_1}{\sin\beta_1} = \frac{\sin\alpha_2}{\sin\beta_2} = \frac{\sin\alpha_3}{\sin\beta_3} \qquad (4.2)$$

Identities 22 and 33 are recognized as the Spherical Law of Cosines. Note that normally the expressions of this law involve the interior vertex angles instead of the supplementary angles that are used here. The other identities have not yet found a use.